1 Limits

Exercise 1.1

Calculate the following limits:

1. $\lim_{x\to 1}(x+2)(x-3)$
2. $\lim_{x\to 3}(x+2)(x-3)$
3. $\lim_{x\to 5}(x+2)(x-3)$
4. $\lim_{x\to 0}\frac{3x+5}{x+2}$
5. $\lim_{x\to 5}\frac{3x+5}{x+2}$
6. $\lim_{x\to -1}\frac{3x+5}{x+2}$
7. $\lim_{x\to 0} 7-9x-x^2$
8. $\lim_{x\to 3} 7-9x-x^2$
9. $\lim_{x\to -1} 7-9x-x^2$
10. $\lim_{x\to 1}\frac{x^2+x-56}{x-7}$
11. $\lim_{x\to 7}\frac{x^2+x-56}{x-7}$
12. $\lim_{x\to 2}\frac{(x+2)^3-8}{x}$
13. $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$
14. $\lim_{x\to 2}\frac{x^2-5x+6}{x^2-x-2}$
15. $\lim_{x\to\infty}(5-\frac{1}{x})$
16. $\lim_{x\to-\infty}(5-\frac{1}{x})$
17. $\lim_{x\to\infty}\frac{100}{x}$
18. $\lim_{x\to\infty}\frac{1}{x^2}$
19. $\lim_{x\to\infty}\frac{5}{\sqrt{x}}$
20. $\lim_{x\to\infty}\frac{7}{9^x}$
21. $\lim_{x\to\infty}\frac{5\cdot 8^x+11}{3\cdot 8^x-1}$
22. $\lim_{x\to\infty}\frac{2\cdot 4^x+5\cdot 3^x-13}{7\cdot 4^x+3}$
23. $\lim_{x\to\infty}\sqrt{x+2}-\sqrt{x}$
24. $\lim_{x\to\infty} x-\sqrt{x^2+7x-1}$

Exercise 1.2

Determine the limits of the function at the borders of the function domain. Then draw a graph of the function.

1. $f(x)=2x-3$
2. $f(x)=-x+2$
3. $f(x)=(x-2)^2-3$
4. $f(x)=-(x+3)^2+2$
5. $f(x)=x^3+2$
6. $f(x)=-(x-1)^3-2$
7. $f(x)=\frac{1}{x}$
8. $f(x)=\frac{1}{x-3}+2$
9. $f(x)=\frac{1}{x^2}-3$
10. $f(x)=-\frac{1}{x^2+2x+1}-3$
11. $f(x)=e^x$
12. $f(x)=e^{x-3}+2$
13. $f(x)=2^{2x-2}+2$
14. $f(x)=(\frac{1}{3})^x+2$
15. $f(x)=(\frac{1}{2})^x-1$
16. $f(x)=\ln x$
17. $f(x)=-\ln x$
18. $f(x)=\ln(x-2)+1$
19. $f(x)=\log_2 x+1$
20. $f(x)=\log_{\frac{1}{3}} x$
21. $f(x)=\log_2(x-1)+2$

2 Definition of a derivative

Exercise 2.1

Find difference quotient, derivative (using the concept of limit) and value of derivative at $x=0$, $x=1$ and $x=3$ for the following functions:

1. $f(x)=x^2-5$
2. $f(x)=-2x^2+7$
3. $f(x)=3x^2-2x$
4. $f(x)=-x^2+x-12$
5. $f(x)=2x^2-3x+4$
6. $f(x)=x^2-4x+3$
7. $f(x)=3x^2-2x+3$
8. $f(x)=-x^2+3x+2$
9. $f(x)=4x^2+5x+2$
10. $f(x)=x^3-17$
11. $f(x)=-x^3+3x^2+3x+6$

3 Formulas for derivatives

$f(x)$	$f'(x)$	Assumptions
a	0	$a \in \mathbb{R}$
x^n	nx^{n-1}	$n \in \mathbb{R}$
e^x	e^x	
a^x	$a^x \ln a$	$a > 0$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\log_a x$	$\frac{1}{x \ln a}$	$a > 0, a \neq 1, x > 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	

$f(x)$	$f'(x)$	Assumptions
$a \cdot g(x)$	$a \cdot g'(x)$	$a \in \mathbb{R}$
$g(x) + h(x)$	$g'(x) + h'(x)$	
$g(x)h(x)$	$g'(x)h(x) + g(x)h'(x)$	
$\frac{g(x)}{h(x)}$	$\frac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$	
$h(g(x))$	$h'(g(x))g'(x)$	

Exercise 3.1

Find derivatives of the following functions:

1. $f(x) = 3x + 1$
2. $f(x) = x^{12} - 2$
3. $f(x) = 63$
4. $f(x) = 7x^5 + 30$
5. $f(u) = 3u^{-1} + 4$
6. $f(u) = -4u^{\frac{1}{2}} - 10$
7. $f(u) = 4u^{\frac{1}{4}} - 1$
8. $f(x) = -x^4 + e$
9. $f(x) = 9x^{\frac{1}{3}}$
10. $f(w) = 5w^4 + 17$
11. $f(x) = cx^2 - 80$
12. $f(x) = ax^b + \pi$
13. $f(x) = ax^{-b} + d$

Exercise 3.2

Find $f'(1)$ and $f'(2)$:

1. $f(x) = 18x + 1$
2. $f(x) = cx^3 - c$
3. $f(x) = -5x^{-2}$
4. $f(x) = \frac{3}{4}x^{\frac{4}{3}} + 17$
5. $f(u) = 6x^{\frac{1}{6}}$
6. $f(u) = -3w^{-\frac{1}{6}}$

Exercise 3.3

Find derivatives of the following functions:

1. $f(x) = 12x^{\frac{7}{4}} + 5x^{-\frac{4}{3}} + 5x^3 + 4x^2 - 3x + 0.67$
2. $f(x) = 6x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} + 2x^4 + 4x^5 + 81x + 112.5$
3. $f(x) = \frac{3}{\sqrt[3]{x^4}} - \frac{2}{\sqrt[5]{x^2}} + \sqrt[-2]{x^7} + \sqrt[7]{x^8} + 5x^3 + 2x - 12.3$
4. $f(x) = \frac{5}{\sqrt[2]{x^5}} - \frac{7}{\sqrt[4]{x^3}} + \sqrt[3]{x^7} + \sqrt[5]{x^3} + 6x^7 + 13x - 11.11$
5. $f(x) = \frac{7}{\sqrt[3]{x^2}} - \frac{3}{\sqrt[7]{x^8}} + \sqrt[-3]{x^4} + \sqrt[3]{x^3} + 3x^5 - 9x + 8.7$
6. $f(x) = 4x^3 - \frac{3}{\sqrt[4]{x^5}} + \sqrt[-4]{x^9} + \sqrt[7]{x^8} + 12x^2 - 4x - 5.5$
7. $f(x) = \frac{-1}{\sqrt[3]{x^5}} - \frac{2}{\sqrt[4]{x^3}} + \sqrt[-7]{x^6} + \sqrt[2]{x^5} + 7x^7 + 3x - 1.32$

Exercise 3.4

Differentiate using the product rule and check the result by first simplifying the formula and then differentiating:

1. $f(x) = x(x - 2)$
2. $f(x) = x^3(2 - x)$
3. $f(x) = (9x^2 - 2)(3x + 1)$
4. $f(x) = (2x - 1)(5x - 2)$
5. $f(x) = (3x + 10)(6x^2 - 7x)$
6. $f(x) = (1 + x^3)(2x^2 - 3)$
7. $f(x) = (x^3 + 3)(2x^2 - 2x)$
8. $f(x) = (2x^4 + 3x)(x^3 - 5x^2)$
9. $f(x) = (2x - x^4)(2x^3 - x^2)$
10. $f(x) = x^2(4x + 6)$
11. $f(x) = (ax - b)(cx^2)$
12. $f(x) = x^3(3 - x^2)(x - 2)$
13. $f(x) = (2 - 3x)(1 + x)(x + 2)$
14. $(2x - 2)(3x - 4)(x2 - 1)(3 + x)$
15. $(5 - x^2)x^5$
16. $(x^2 + 3)x^{-1}$
17. $(x^3 + 1)x^{-7}$
18. $(2x + x^4)x^{-3}$

Exercise 3.5

Find the derivatives using quotient rule:

1. $\frac{(x^2+3)}{x}$
2. $\frac{(x+9)}{x}$
3. $\frac{6x}{x+5}$
4. $\frac{3+x}{1-x}$
5. $\frac{9x^2-2}{3x+1}$
6. $\frac{2x-1}{5x-2}$
7. $\frac{3x+10}{6x^2-7x}$
8. $\frac{x^3+3}{2x^2-2x}$
9. $\frac{2x^4+3x}{x^3-5x^2}$
10. $\frac{2x-x^4}{2x^3-x^2}$
11. $\frac{1+x^3}{2x^2-3}$
12. $\frac{5-x^2}{x^5}$
13. $\frac{x^3+1}{x^7}$
14. $\frac{2x+x^4}{x^{-3}}$

Exercise 3.6

Draw the following functions using first and second derivatives as your guide:

1. $f(x) = x^2 - 3x + 3$
2. $f(x) = -x^2 + 2x - 4$
3. $f(x) = x^3 - 3x + 9$
4. $f(x) = -x^3 + 2x + 8$
5. $f(x) = x^3 - 3x^2$
6. $f(x) = 3x^5 - 5x^4$
7. $f(x) = x^4 - 10x^2 + 9$
8. $f(x) = x^5 + 2x^2$
9. $f(x) = x^3 - 3x^2$

Exercise 3.7

Use chainrule to find the derivative of:

1. $y = u^3$, where $u = 5 - x^2$
2. $w = ay^2$, where $y = bx^2 + cx$
3. $z = y^2$, where $y = 2x + 5$
4. $z = (1/2)y^2 - 3$, where $y = 3x^2$
5. $y = 2u^4 + 3$, where $u = 5 - 3x^3$
6. $w = 5y + 4$, where $y = ax^2 - cx$
7. $z = y^2$, where $y = 2x + 5$
8. $z = y^3$, where $y = x^3$

Exercise 3.8

Use the chain rule to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

1. $y = \left(3x^2 - 13\right)^3$
2. $y = \left(7x^3 - 5\right)^9$
3. $y = (ax + b)^5$
4. $y = \left(2x^3 + 3x^2 - 7x + 9\right)^{70}$
5. $y = \left(3x^4 + 2x^7 - 2x + 3\right)^{11}$
6. $y = \left(5x^4 + x^5 + 3x + 4\right)^{31}$
7. $y = \left[(-2x + 3)^7 + 4\right]^{12}$
8. $y = \left[(x + 1)^{-3} + 6\right]^{10}$

Exercise 3.9

Use product or quotient and the chain rule to calculate derivatives of the following:

1. $(x - 1)^{10}(x + 2)^{12}$
2. $\frac{(x-1)^{10}}{(x+2)^{12}}$
3. $\left(x^2 - 3\right)^{100}\left(4x^3 + 2x\right)^{10}$
4. $\frac{\left(x^2-3\right)^{100}}{(4x^3+2x)^{10}}$
5. $(ax + b)^c(d - x)^{10}$
6. $\frac{(ax+b)^c}{(d-x)^{10}}$
7. $\left(7x^3 - 5\right)^9\left(3x^2 - 13\right)^3$
8. $\frac{\left(3x^2-13\right)^3}{\left(3x^2-13\right)^3}$

Exercise 3.10

For $y = 6x + 36$, find inverse function and find $\frac{\mathrm{d}x}{\mathrm{d}y}$.

Exercise 3.11

Given y, find its inverse function. Then find both $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}y}$ and verify the inverse function rule. Check that the graphs of the two functions bear a mirror-image relationship to each other.

1. $y = 7x + 21$
2. $y = 0.5x + 5$

Exercise 3.12

Find derivatives of:

1. $y = 4e^x$
2. $y = 5^x$
3. $y = e^{2t+4}$
4. $y = e^{1-9t}$
5. $y = 3^x$
6. $y = 5^{3x\hat{}2-e}$
7. $y = 2{*}2^x$
8. $y = 13^{2x+3}$
9. $y = e^{\left(t^2 2\right)+1}$
10. $y = 5e^{2-\left(t\hat{}2\right)}$
11. $y = e^{a\left(x\hat{}2\right)+bx+c}$
12. $y = 7e^{2x\hat{}3+2x-89}$
13. $y = 2e^{7x\hat{}4-8x-125}$
14. $y = xe^x$
15. $y = x^2e^{2x}$
16. $y = axe^{bx+c}$
17. $y = 3x2^{x+1}$
18. $y = 7x(\frac{1}{2})^{x^{12+2x-17}}$
19. $y = x^3(\frac{5}{7})^{2x^{3+5x-24}}$
20. $y = e^{x-3}(\frac{1}{3})^{x^{5-7x-34}}$

Exercise 3.13

Find derivatives of:

1. $y = \ln\left(7x^5\right)$
2. $y = \ln\left(ax^c\right)$
3. $y = \ln(x + 19)$
4. $y = \ln\left(7x^3 + 2x + 1\right)$
5. $y = 5\ln(t + 1)^2$
6. $y = \ln\left[\left(3x^2 - 13\right)^3\right]$
7. $y = \ln(x) - \ln(1 + x)$
8. $y = \ln\left[x(1 - x)^8\right]$
9. $y = \ln\left[\frac{2x}{1+x}\right]$
10. $y = 5x^4\ln\left(x^2\right)$
11. $y = 9x^3\ln\left(2x^3 + 7x^2 - 3x + 12\right)$
12. $y = 3x^4\ln\left(3x^2 + 2x + 66\right)$

Exercise 3.14

Find derivatives of:

1. $y = \log_2\left(3x^2 - 2x + 3\right)$
2. $y = \log_2(x + 1)$
3. $y = \log_7\left(7x^2 + 17x\right)$
4. $y = \log_2\left(8x^2 + 3x - 12\right)$
5. $y = \log_2\left(-6x^3 - x + 4\right)$
6. $y = 9x^3\ln\left(7x^{3,5} - 5x^2 - 5x + 8\right)$
7. $y = x2\log 3(x + 1)$

Exercise 3.15

Find derivatives of the following functions:

1. $y = \frac{e^x}{(x+4)}$
2. $y = \frac{2e^{3x}}{\ln(2-x)}$
3. $y = \frac{x^3-2x^2+3x}{\ln(2-x)}$
4. $y = (2x - 3)e^{3x^3}$
5. $y = \left(x^2 + 3\right)e^{x^2+1}$
6. $y = x^ae^{kx-c}$
7. $y = dx^{-a}e^{kx^b-c}$
8. $y = \frac{7e^{3x^3+2x-7}}{\ln(3x-2x^2)}$
9. $y = \frac{ae^{bx^c+dx}}{\ln(k-x^2)}$

Exercise 3.16

Draw the following functions using first and second derivatives as your guide:

1. $y = x^2 - 3x + 3$
2. $y = -x^2 + 2x - 4$
3. $y = x^3 - 3x + 9$
4. $y = -x^3 + 2x + 8$
5. $y = x^3 - 3x + 9$
6. $y = x^3 - 3x^2$
7. $y = 3x^5 - 5x^4$
8. $y = x^4 - 10x^2 + 9$
9. $y = x^5 + 2x^2$
10. $y = x^3 - 3x^2$
11. $y = 5x^5 - 2x^3 - 3x^2$
12. $y = 2x^5 + 1,5x^3 - 5x^2$
13. $y = \frac{x-1}{x-2}$
14. $y = \frac{x^2}{x-1}$
15. $y = \frac{x+2}{(x-1)^2}$
16. $y = e^{-\frac{1}{2}x^2}$
17. $y = \ln\left(x^2 - 1\right)$
18. $y = xe^{-x}$
19. $y = x\ln x$

Exercise 3.17

Find the differential dy for:

1. $y = x^3 - 3x^2$
2. $y = e^{2x} - \ln\left(x^2 + 2\right)$
3. $y = 4x^5 - 3x^4y + 4x^3 - 3x^2$

4 One Variable Function Optimisation

Exercise 4.1

Let q denote the level of production in your company. Moreover, let p, r and c be functions of q which yields the price of the product, total revenue and total costs of the company, respectively. Find optimal level of production in the following cases:

1. $$\begin{cases} p(q) = 520 - 3q \\ c(q) = \frac{1}{2}q^2 + 100q + 4000 \end{cases}$$

2. $$\begin{cases} r(q) = -\frac{1}{3}q^2 + 4000q \\ c(q) = 2000q + 1000000 \end{cases}$$

3. $$\begin{cases} p(q) = 90 - 8q \\ c(q) = 10q + 150 \end{cases}$$

4. $$\begin{cases} p(q) = -\frac{1}{2}q + 12 \\ c(q) = \frac{1}{4}q^2 + 3q \end{cases}$$

5. $$\begin{cases} p(q) = -300q + 9000 \\ c(q) = 200q^2 + 1000q + 30000 \end{cases}$$

6. $$\begin{cases} p(q) = -\frac{2}{100}q + 500 \\ c(q) = \frac{3}{100}q^2 + 150q + 12500 \end{cases}$$

Exercise 4.2

Find fix cost function, average cost function and marginal cost function, when total cost function is given by:

1. $\mathrm{C} = \mathrm{Q}^3 - 4\mathrm{Q}^2 + 10\mathrm{Q} + 75$
2. $\mathrm{C} = \mathrm{Q}^3 - 5\mathrm{Q}^2 + 12\mathrm{Q} + 100$
3. $C = Q^2 - 4Q + 174$
4. $C = Q^3 - 3Q^2 + 15Q + 200$

Exercise 4.3

Find the marginal and average functions for the following total functions and graph the results:

1. $\mathrm{C} = 3\mathrm{Q}^2 + 7\mathrm{Q} + 12$
2. $R = 10Q - Q^2$
3. $Q = aL + bL^2 - c^3 (a, b, c > 0)$

Exercise 4.4

Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MMP_K a function of K alone, or both K and L ? What about MP_L ?

5 Optimisation of Multivariate Functions

Exercise 5.1

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (sometimes denoted as f_x and f_y, respectively) from the following:

1. $f(x, y) = 2x^3 - 11x^2y + 3y^2$
2. $f(x, y) = 7x^3 + 6xy^2 - 9y^3$
3. $f(x, y) = (2x + 3)(y - 2)$
4. $f(x, y) = \frac{5x+3}{y-2}$
5. $f(x, y) = x^2 + 5xy - y^3$
6. $f(x, y) = \left(x^2 - 3y\right)(x - 2)$
7. $f(x, y) = \frac{2x-3y}{x+y}$
8. $f(x, y) = \frac{x^2-1}{xy}$

Exercise 5.2

Find local minimums and maximums of the following functions:

a) $f(x, y) = x^2 + xy + 2y^3 + 3$

b) $f(x, y) = -x^2 + y^2 + 6x + 2y$

c) $f(x, y) = x + 2ey - e^x - e^{2y}$

d) $f(x, y) = e^{2x} - 2x + 2y^2 + 3$

e) $f(x, y, z) = x^2 + 3y^2 - 3xy + 4yz + 6z^2$

f) $f(x, y, z) = 29 - (x^2 + y^2 + z^2)$

g) $f(x, y, z) = xz + x^2 - y + yz + y^2 + 3z^2$

h) $f(x, y, z) = e^{2x} + e^{-y} + e^{z^2} - (2x + 2e^z - y)$

Exercise 5.3

Let x and y denote the levels of production for two goods X and Y. Moreover, let π denote the function which yields profit based on production levels. Find an optimal level of production in the following cases:

a) $\pi(x,y) = -2x^2 - y^3 + 6x + 12y$ b) $\pi(x,y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$

c) $\pi(x,y) = \frac{-x^2y + xy^2 - 1}{3y^3 + 9y}$ d) $\pi(x,y) = 8x + 10y - \frac{1}{1000}(x^2 + xy^2 + y^2) - 10000$

Exercise 5.4

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $p_x = 12$ and $p_y = 18$ be the prices of the products X and Y respectively. Finally, let $c(x,y) = 2x^2 + xy + 2y^2 + 24$ denote the function which yields total costs based on production levels. Find an optimal level of production.

Exercise 5.5

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $p_x = p_y = 150$ be the prices of the products X and Y respectively. Finally, let $c_x(x) = \frac{2}{1000}x^2 + 4x + 456000$ and $c_y(y) = \frac{5}{1000}y^2 + 4y + 274000$ denote functions which yield total costs based on production levels for products X and Y, respectively. Find an optimal level of production.

Exercise 5.6

Let x and y denote the levels of production for two goods X and Y that your company is selling. Moreover, let $r(x,y) = -\frac{5}{1000}x^2 - \frac{3}{1000}y^2 - \frac{2}{1000}xy + 20x + 17y$ and $c(x,y) = 6x + 3y + 1000$ denote functions of production levels which yield total revenue and total costs, respectively. Find an optimal level of production.

Exercise 5.7

Find extremums of function f with respect to given restrictions:

a) $\begin{cases} f(x,y) = xy + 2x \\ 4x + 2y = 60 \end{cases}$ b) $\begin{cases} f(x,y) = xy \\ 4x + 2y = 40 \end{cases}$

Exercise 5.8

A company CoalMeMaybe (CMM) has access to two power plants A and B. If we denote by x and y tons of coal used in power plant A and B, respectively, then the cost functions can be expressed as $c_A(x) = 2(x-1)$ and $c_B(y) = (y-3)^2$ in A and B respectively. From one ton of coal it is possible to produce 5MWh at A and 2MWh at B. CMM is supposed to product 100 MWh. What division of resources between two power plants will be optimal?

Exercise 5.9

Denote by a, b and t the cost of the hour labour, the cost of unit capital and the total budget for the project, respectively. Moreover, denore by q the function which yiedls level of production based on level of labor l and capital k. Find level of labor and capital which maximizes level of production when:

a) $\begin{cases} a = 3, b = 5, t = 1200 \\ q(l,k) = 6l + 10k + lk \end{cases}$ b) $\begin{cases} a = 6, b = 3, t = 69 \\ q(l,k) = 5l + 2k + kl \end{cases}$

a) $\begin{cases} a = 2, b = 1, t = 99 \\ q(l,k) = 2\sqrt{l} + 3\sqrt{k} \end{cases}$ b) $\begin{cases} a = 2, b = 3, t = 1200 \\ q(l,k) = 50lk \end{cases}$